Math 247A Lecture 10 Notes

Daniel Raban

January 29, 2020

1 Characterization of A_p Weights

1.1 A_p weights for $p > 1$

Last time, we began to tackle the problem of characterizing nonnegative measures μ for which

$$\int |Mf(y)|^p \, d\mu(y) \lesssim |f(y|^p \, d\mu(p).$$

uniformly for $f \in L^p(d\mu)$ and some $1 < p < \infty$. We will not prove the full details, but we will give a compelling intuition of the results.

Fix $1 < p < \infty$. Recall that a locally integrable weight $\omega : \mathbb{R}^d \to [0, \infty)$ satisfies the A_p **condition** if there is an $A > 0$ such that

$$\sup_{\text{balls } B} \frac{1}{|B|} \int_B \omega(y) \, dy \left[\frac{1}{|B|} \int_B \omega^{-p'/p}(y) \, dy \right]^{p/p'} \leq A.$$

This is equivalent to

$$\sup_{\text{balls } B} |B|^{-p} \omega(B) \|\omega^{-1/(p-1)}\|_{L^1(\mathbb{R})}^{p-1} \leq A.$$

Remark 1.1. If $1 < p < q < \infty$, then $A_p \subseteq A_q$: Let $\omega \in A_p$. Then by Hölder,

$$\begin{aligned} \|\omega^{-1/(q-1)}\|_{L^1(B)} &\leq \|\omega^{-1/q-1}\|_{L^{(q-1)/(p-1)}} |B|^{1-(p-1)/(q-1)} \\ &= \|\omega^{-1/(p-1)}\|_{L^1(B)}^{(p-1)/(q-1)} |B|^{(q-p)/(q-1)}. \end{aligned}$$

So we get

$$|B|^{-q} \omega(B) \|\omega^{-1/(q-1)}\|_{L^1(B)}^{q-1} \leq |B|^{-p} \omega(B) \|\omega^{-1/(p-1)}\|_{L^1(B)}^{p-1} \lesssim 1$$

uniformly in B.

Lemma 1.1. *Fix $1 \le p < \infty$. Then $\omega \in A_p$ if and only if*

$$\left[\frac{1}{|B|}\int_B f(y)\,dy\right]^p \lesssim \frac{1}{\omega(B)}\int_B |f(y)|^p \omega(y)\,dy,$$

uniformly in $f \ge 0$ and balls B.

We proved this last time for $p = 1$.

Proof. It remains to consider $1 < p < \infty$.
($\implies$):

$$\begin{aligned}\left[\frac{1}{|B|}\int_B f(y)\,dy\right]^p &= \left[\frac{1}{|B|}\int_B f(y)\omega^{1/p-1/p}\,dy\right]^p \\ &\le |B|^{-p}\int |f(y)|^p\omega(y)\,dy \int_B \underbrace{\left[\omega(y)^{-p'/p}\,dy\right]^{p/p'}}_{\lesssim |B|^p \cdot 1/\omega(B)} \\ &\lesssim \frac{1}{\omega(B)}\int_B |f(y)|^p\omega(y)\,dy.\end{aligned}$$

($\impliedby$): Fix $\varepsilon > 0$ and a ball B. Let $f = (\omega+\varepsilon)^{-p'/p}$. Then

$$\begin{aligned}\left[\frac{1}{|B|}\int_B (\omega+\varepsilon)^{-p'/p}(y)\,dy\right]^p &\lesssim \frac{1}{\omega(B)}\int_B (\omega+\varepsilon)^{-p'}(y)\omega(y)\,dy \\ &\lesssim \frac{1}{\omega(B)}\int (\omega+\varepsilon)^{-p'+1}(y)\,dy\end{aligned}$$

Note that $p' - 1 = -p'/p$.

$$\lesssim \frac{1}{\omega(B)}\int_B (\omega+\varepsilon)^{-p'/p}(y)\,dy.$$

So

$$|B|^{-p}\omega(B)\left[\int_B (\omega+\varepsilon)^{-p'/p}(y)\,dy\right]^{p/p'} \lesssim 1$$

uniformly in B and $\varepsilon > 0$. Let $\varepsilon \to 0$ and use the monotone convergence theorem □

Corollary 1.1. *Fix $1 \le p < \infty$. If $\omega \in A_p$, then ω is a doubling measure.*

Proof. Let $B = B(x, 2r)$ and $f = \mathbb{1}_{B(x,r)}$. Then

$$\left[\frac{|B(x,r)|}{|B(x,2r)|}\right]^p \lesssim \frac{\omega(B(x,r))}{\omega(B(x,2r))}$$

uniformly in $x \in \mathbb{R}^d$ and $r > 0$. □

Remark 1.2. In fact, A_p weights with $1 \le p < \infty$ satisfy a "fairness condition": If $F \subseteq B$, then taking $f = \mathbb{1}_F$, we get

$$\left[\frac{|F|}{|B|}\right]^p \lesssim \frac{\omega(F)}{\omega(F)}.$$

So if F is a large chunk of the ball B, ω has to give a large proportion of the measure of the ball to F; it has to treat F fairly.

1.2 Proof sketch of characterization

Theorem 1.1. *Fix $1 < p < \infty$. Then $\omega \in A_p$ if and only if*

$$\int |Mf(y)|^p \omega(y)\,dx \lesssim \int |f(y)|^p \omega(y)\,dy$$

uniformly for $f \in L^p(\omega\,dx)$ (that is, $M : L^p(\omega\,dx) \to L^p(\omega\,dx)$ boundedly).

This answers the question we proposed but only in the case that ω is a weight; i.e. ω is absolutely continuous with respect to Lebesgue measure.

Remark 1.3. ($\Longleftarrow$): The necessity holds under even weaker assumptions. If $M : L^p(\omega\,dx) \to L^{p,\infty}(\omega\,dx)$ boundedly, then $\omega \in A_p$.

Proof. ($\Longleftarrow$): Fix a ball B of radius $r > 0$. Fix $\varepsilon > 0$, and let $f = (\omega + \varepsilon)^{-p'/p}\mathbb{1}_B$. For $x \in B$,

$$\begin{aligned}
Mf(x) &= \sup_{R>0} \frac{1}{|B(x,R)|}\int_{B(x,R)} (\omega+\varepsilon)^{-p'/p}\mathbb{1}_B(x)\,dx \\
&\ge \frac{1}{|B(x,2r)|}\int_B (\omega+\varepsilon)^{-p'/p}(y)\,dy \\
&= \frac{1}{2^d|B|}\int_B (\omega+\varepsilon)^{p'/p}(y)\,dy
\end{aligned}$$

Let's give this a name

$$=: 2\lambda.$$

Now

$$\begin{aligned}
\omega(B) &\le \omega(\{x : Mf(x) > \lambda\}) \\
&\lesssim \frac{\int_B (\omega+\varepsilon)^{-p'}(y)\omega(y)\,dy}{\lambda^p} \\
&\lesssim \frac{\int_B (\omega+\varepsilon)^{-p'/p}(y)\,dy}{[\int_B (\omega+\varepsilon)^{-p'/p}(y)\,dy]^p}.
\end{aligned}$$

So we get that

$$|B|^{-p}\omega(B)\left[\int_B (\omega+\varepsilon)^{p'/p}(y)\,dy\right]^{p/p'} \lesssim 1. \qquad \square$$

Remark 1.4. ($\implies$): The sufficiency, that is, $\omega \in A_p \implies M$ is bounded on $L^p(\omega\, dx)$, rests on three ingredients:

1. If $1 \le q < \infty$, then $M : L^q(\omega\, dx) \to L^{q,\infty}(\omega\, dx)$ boundedly (this is Homework exercise 10). Look at
$$M_\omega f(x) = \sup_{r>0} \frac{1}{\omega(B(x,r))} \int_{B(x,r)} |f(y)|\omega(y)\, dy.$$
Then $M_\omega : L^1(\omega\, dx) \to L^{1,\infty}(\omega\, dx)$ boundedly. Then
$$\left| \frac{1}{|B|} \int_B f(y)\, dy \right|^p \lesssim \frac{1}{\omega(B)} \int_B |f(y)|^p \omega(y)\, dy,$$
which tells us
$$|Mf|^p \lesssim M_\omega(|f|^p).$$

2. (Appears in Ch5 of Stein's Harmonic Analysis textbook[1]) A reverse Hölder inequality: If $\omega \in A_\infty = \bigcup_{1 \le p < \infty} A_p$, then there exist and $r > 1$ and $c > 0$ (both depending on ω) such that
$$\left[\frac{1}{|B|} \int_B \omega^r(y)\, dy \right]^{1/r} \le \frac{c}{|B|} \int_B \omega(y)\, dy \iff |B|^{1/r'} \|\omega\|_{L^r(B)} \le c\|\omega\|_{L^1(B)}.$$
This implies that if $\omega \in A_p$, then $\omega \in A_q$ for some $q < p$.

Ingredients 1 and 2 give $M : L^q(\omega\, dx) \to L^{q,\infty}(\omega\, dx)$ boundedly for some $q < p$.

3. The Marcinkiewicz interpolation theorem with $M : L^\infty(\omega\, dx) \to L^\infty(\omega\, dx)$ (use the fact that $|E| = 0 \iff \omega(E) = 0$ since $\omega > 0$ a.e. as $\omega \in A_p$).

Next time, we will discuss how this generalizes to arbitrary measures, not just ones absolutely continuous with respect to Lebesgue measure.

[1] This book is the bible of Harmonic Analysis.